

Lecture 20

April 29th, 2004

Difference Quotients and Sobolev spaces

Define

$$\Delta_i^h u := \frac{u(x + h \cdot \mathbf{e}_i) - u(x)}{h}, \quad h \neq 0.$$

Lemma. Let Ω be a bounded domain in \mathbb{R}^n , and $u \in W^{1,p}(\Omega)$, for some $1 \leq p < \infty$. Then for any $\Omega' \subset \Omega$ such that $\text{dist}(\Omega', \partial\Omega) > h$ holds

$$\|\Delta_i^h u\|_{L^p(\Omega')} \leq \|D_i u\|_{L^p(\Omega)}.$$

Proof.

$$\begin{aligned} |\Delta_i^h u| &= \left| \frac{u(x + h \cdot \mathbf{e}_i) - u(x)}{h} \right| \leq \frac{1}{h} \int_0^h |D_i u(x_1, \dots, x_i + \zeta, \dots, x_n)| d\zeta \\ &\leq \frac{1}{h} \left\{ \int_0^h 1^q \right\}^{\frac{1}{q}} \left\{ \int_0^h |D_i u(x_1, \dots, x_i + \zeta, \dots, x_n)|^p d\zeta \right\}^{\frac{1}{p}} \Rightarrow \\ |\Delta_i^h u|^p &\leq h^{\frac{p}{q}-p} \cdot \int_0^h |D_i u(x_1, \dots, x_i + \zeta, \dots, x_n)|^p d\zeta \\ &= \frac{1}{h} \cdot \int_0^h |D_i u(x_1, \dots, x_i + \zeta, \dots, x_n)|^p d\zeta \Rightarrow \\ \int_{\Omega'} |\Delta_i^h u|^p &\leq \frac{1}{h} \cdot \int_{\Omega'} \int_0^h |D_i|^p d\zeta d\mathbf{x} = \frac{1}{h} \cdot \int_0^h \int_{\Omega} |D_i|^p d\mathbf{x} d\zeta \\ &= \frac{1}{h} \int_0^h \|D_i u\|_{L^p(\Omega')}^p = \|D_i u\|_{L^p(\Omega')}^p \leq \|D_i u\|_{L^p(\Omega)}^p, \end{aligned}$$

where we applied Fubini's Theorem in order to switch order of integration. ■

Conversely we have

Lemma. Let $u \in L^p(\Omega)$ for some $1 \leq p < \infty$ and suppose $\Delta_i^h u \in L^p(\Omega')$ with $\|\Delta_i^h u\|_{L^p(\Omega')} \leq K$ for all $\Omega' \subset \Omega$ and $0 < |h| < \text{dist}(\Omega', \Omega)$. Then the weak derivative satisfies $\|D_i u\|_{L^p(\Omega)} \leq K$.

Consequently if this holds for all $i = 1, \dots, n$ then $u \in W^{1,p}(\Omega)$.

Proof. We will make use of

Alouglou's Theorem. A bounded sequence in a separable, reflexive Banach space has a weakly convergent subsequence.

A topological space is called *separable* if it contains a countable dense set.

A Banach space is called *reflexive* if $(B^*)^* = B$.

A sequence $\{x_n\}$ in a Banach space is said to *converge weakly* to x when $\lim_{n \rightarrow \infty} F(x_n) \rightarrow F(x)$ for all linear functionals $F \in B^*$. This is sometimes denoted $\lim_{n \rightarrow \infty} x_n \rightharpoonup x$.

Example: Let $\ell^2 := \left\{ (a_1, a_2, \dots) : \sum_{i=1}^{\infty} a_i^2 < \infty \right\}$. Consider the sequence $\{x_i := (0, \dots, 0, 1, 0, \dots)\}$

$\subseteq \ell^2$. Any bounded linear functional on ℓ^2 will be some linear combination of the linear functionals F_j , defined by $F_j(a_1, \dots) = a_j$ (each such linear combination corresponds exactly to a point in ℓ^2).

That makes sense, indeed by the Riesz Representation Theorem $(\ell^2)^* = \ell^2$ (note ℓ^2 is a Hilbert space not just a Banach space as it has an inner product structure). For any such $F = (a_1, \dots)$,

$\lim_{i \rightarrow \infty} F(x_i) = \lim_{i \rightarrow \infty} a_i = 0$. So x_i converges to the 0 vector weakly, though certainly not strongly:

by Fourier Theory each point in ℓ^2 corresponds to a periodic function on $[0, 1]$, i.e an element of

$L^2(S^1)$, and of course $\lim_{n \rightarrow \infty} \exp(n2\pi\sqrt{-1}z) \not\rightarrow 0(z)$.

We come back to the proof. For the Banach space $B = L^p(\Omega)$, $B^* = L^q(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

This can be seen directly: If $F \in (L^p(\Omega))^*$, then exists f such that $F(g) = \int_{\Omega} g \cdot f$, $\forall g \in L^p(\Omega)$, and this will be bounded iff $f \in L^q(\Omega)$. So we get an identification $F \in (L^p(\Omega))^* \cong L^q(\Omega)$.

By Alouglou's Theorem there exists a sequence $\{h_m\} \rightarrow 0$ with $\Delta_i^{h_m} u \rightarrow v \in L^p(\Omega)$. In other words

$$\int_{\Omega} \psi \cdot \Delta_i^{h_m} u \rightarrow \int_{\Omega} \psi \cdot v \in L^p(\Omega), \quad \forall \psi \in L^q(\Omega).$$

And in particular for any $\psi \in \mathcal{C}_0^1(\Omega)$ (which is dense in $L^q(\Omega)$ so will suffice to look at such ψ as will become clear ahead)

$$\begin{aligned} \int_{\Omega} \psi \Delta_i^{h_m} u &= \int_{\Omega} \psi \frac{1}{h} (u(x + h \cdot \mathbf{e}_i) - u(x)) d\mathbf{x} \\ &= \frac{1}{h} \int_{\Omega} \psi(x - h \mathbf{e}_i) u(x) d\mathbf{x} - \frac{1}{h} \int_{\Omega} \psi(x) u(x) d\mathbf{x} \\ &= \int_{\Omega} \frac{1}{h} (\psi(x - h \mathbf{e}_i) - \psi(x)) u(x) d\mathbf{x} \\ &= \int_{\Omega} -\Delta_i^h \psi(x) u(x) d\mathbf{x} \xrightarrow{h \rightarrow 0} \int_{\Omega} -D_i \psi(x) u(x) d\mathbf{x} \end{aligned}$$

since ψ is continuously differentiable. Altogether

$$\int_{\Omega} \psi \cdot v \in L^p(\Omega) = \int_{\Omega} -D_i \psi(x) u(x) d\mathbf{x},$$

which by definition means v is the weak derivative of u in the direction of the x_i axis, or simply the undistinctive notation $v = D_i u$.

We also get the desired estimate, using the Fatou Lemma $\int \liminf \leq \liminf \int$

$$\int_{\Omega} |D_i u|^p d\mathbf{x} = \int_{\Omega} \liminf |\Delta_i^h u|^p d\mathbf{x} \leq \liminf \int_{\Omega} |\Delta_i^h u|^p d\mathbf{x} \leq K^p,$$

i.e $\|D_i u\|_{L^p(\Omega)} \leq K$. ■

L^2 Theory

Consider the second order equation in divergence form

$$Lu \equiv L(u) := D_i(a^{ij}D_j u) + b^i D_i u + c \cdot u = f,$$

with $a^{ij}, b^i, c \in L^1(\Omega)$ (integrable coefficients).

We call $u \in W^{1,2}(\Omega)$ a *weak solution* of the equation if

$$\forall v \in \mathcal{C}_0^1(\Omega) \quad - \int_{\Omega} a^{ij} D_j u D_i v + \int_{\Omega} (b^i D_i u + c u) v = \int_{\Omega} f v.$$

Elliptic Regularity

Let $u \in W^{1,2}(\Omega)$ be a weak solution of $Lu = f$ in Ω , and assume

- L strictly elliptic with $(a^{ij}) > \gamma \cdot I$, $\gamma > 0$
- $a^{ij} \in \mathcal{C}^{0,1}(\Omega)$
- $b^i, c \in L^\infty(\Omega)$
- $f \in L^2(\Omega)$

Then for any $\Omega' \subset \Omega$, $u \in W^{2,2}(\Omega')$ and

$$\|u\|_{W^{2,2}(\Omega')} \leq C(\|a^{ij}\|_{C^{0,1}(\Omega)}, \|b\|_{C^0(\Omega)}, \|c\|_{C^0(\Omega)}, \lambda, \Omega', \Omega, n) \cdot (\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)}).$$

Note $L^\infty(\Omega)$ stands for bounded functions on Ω while $\mathcal{C}^0(\Omega)$ are functions that are also continuous (Ω being bounded).

Proof. Start with the definition of u being a solution in the weak sense, $\forall v \in \mathcal{C}_0^1(\Omega)$:

$$\int_{\Omega} a^{ij} D_j u D_i v = \int_{\Omega} (b^i D_i u + c - f) v.$$

and take difference quotients, that is replace v with $\Delta^{-h}v$.

$$\int_{\Omega} a^{ij} D_j u D_i (\Delta^{-h}v) = \int_{\Omega} (b^i D_i u + c - f) (\Delta^{-h}v).$$

Taking $-h$ is a technicality that will unravel its reason later on, and we really mean $\Delta_k^{-h}v$ for some $k \in \{1, \dots, n\}$ and then eventually repeat the computation for all k in that range. This will become clear as well. Finally our goal will be to use the Chain Rule and move the difference quotient operator onto u under the integral sign and get uniform bounds on $\Delta^h Du$ and in this way get a priori $W^{2,2}(\Omega)$ estimates.

The Chain Rule gives

$$\begin{aligned} \Delta^h(a^{ij} D_j u) &= \\ \frac{1}{h} &\left(a^{ij} u(x + h \cdot \mathbf{e}_k) D_j u(x + h \cdot \mathbf{e}_k) - \{a^{ij}(x) - a^{ij}(x + h \cdot \mathbf{e}_k) + a^{ij}(x + h \cdot \mathbf{e}_k)\} D_j u(x) \right) \\ &= a^{ij} u(x + h \cdot \mathbf{e}_k) \Delta^h D_j u - \Delta^h a^{ij} D_j u. \end{aligned}$$

And applied to our previous equation, a short calculation verifies that we can 'integrate by part' WRT $\Delta^h -$

$$\begin{aligned} \int_{\Omega} a^{ij} D_j u D_i (\Delta^{-h}v) &= \int_{\Omega} \Delta^h(a^{ij} D_j u) D_i v \Rightarrow \\ \int_{\Omega} a^{ij} u(x + h \cdot \mathbf{e}_k) \Delta^h D_j u D_i v &= \int_{\Omega} -\Delta^h a^{ij} D_j u D_i v + \int_{\Omega} (b^i D_i u + c - f) (\Delta^{-h}v) \Rightarrow \\ \left| \int_{\Omega} a^{ij} u(x + h \cdot \mathbf{e}_k) \Delta^h D_j u D_i v \right| &\leq \|\Delta^h a^{ij} D_j u\|_{L^2(\Omega)} \|D_i v\|_{L^2(\Omega)} + \\ &\quad + \|b^i D_i u + c - f\|_{L^2(\Omega)} \|\Delta^{-h}v\|_{L^2(\Omega)}, \end{aligned}$$

where we have used the Hölder Inequality for $p = q = 2$. This in turn can be bounded by

$$\begin{aligned} &\leq \|a^{ij}\|_{C^{0,1}(\Omega)} \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \\ &\quad + (\|b^i\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\Omega)} + \|c\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}) \|\nabla v\|_{L^2(\Omega)} \\ &\leq C(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)}) \cdot \|\nabla v\|_{L^2(\Omega)}. \end{aligned}$$

where we have used the Hölder Inequality for $p = 1, q = \infty$, i.e a simple bounded integration argument (e.g $\|cu\|_{L^2(\Omega)} = (\int c^2 \cdot |u|^2)^{\frac{1}{2}} \leq (\sup |c|^2 \int_O |u|^2)^{\frac{1}{2}}$), and $\Delta^h a^{ij} \rightarrow D_k a^{ij}$ as $a^{ij} \mathcal{C}^{0,1}(\Omega)$.

Take a cut-off function $\eta \in \mathcal{C}_0^1(\Omega)$, $0 \leq |\eta| \leq 1$, $\eta|_{\Omega'} = 1$. We now choose a special v : $v := \eta^2 \Delta^h u$. From uniform ellipticity ($a^{ij} \zeta_i \zeta_j \geq \lambda |\zeta|^2$)

$$\lambda \int_\Omega |\eta \nabla \Delta^h u|^2 \leq \int_\Omega \eta^2 a^{ij}(x + h \cdot \mathbf{e}_k) D_i \Delta^h u D_j \Delta^h u.$$

Now

$$D_i v = 2\eta D_i \eta \Delta^h u + \eta^2 D_i \Delta^h u$$

which we substitute into our previous inequality,

$$\begin{aligned} \int_\Omega \eta^2 a^{ij}(x + h \cdot \mathbf{e}_k) D_j \Delta^h u D_i \Delta^h u &\leq \int_\Omega a^{ij}(x + h \cdot \mathbf{e}_k) D_j \Delta^h u \cdot (D_i v - 2\eta D_i \eta \Delta^h u) \\ &\leq C(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)}) \|\nabla v\|_{L^2(\Omega)} + \\ &\quad + C' \|(D \Delta^h u) \eta\|_{L^2(\Omega)} \|D \eta \Delta^h u\|_{L^2(\Omega)} \end{aligned}$$

again by the Hölder Inequality. Now since $\eta \leq 1$

$$\|D_i v\|_{L^2(\Omega)} \leq C'' (\|D_i \eta \Delta^h u\|_{L^2(\Omega)} + \|D \Delta^h u\|_{L^2(\Omega)}).$$

Combining all the above and again using $\eta \leq 1$,

$$\begin{aligned}
\lambda \int_{\Omega} |\eta D\Delta^h u|^2 &\leq C(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)}) \cdot C''(\|D\eta\Delta^h u\|_{L^2(\Omega')} + \|D\Delta^h u\|_{L^2(\Omega')}) \\
&\quad + C' \|(\Delta^h u)\|_{L^2(\Omega')} \|D\eta\Delta^h u\|_{L^2(\Omega')} \\
&\leq c(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)} + \|D\eta\Delta^h u\|_{L^2(\Omega')}) \cdot \|(\Delta^h u)\|_{L^2(\Omega')} \\
&\quad + c(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)}) \cdot \|D\eta\Delta^h u\|_{L^2(\Omega')}.
\end{aligned}$$

Using the AM-GM Inequality $ab = \sqrt{\frac{1}{\epsilon}a^2 \cdot \epsilon b^2} \leq \frac{1}{2}(\frac{1}{\epsilon}a^2 + \epsilon b^2)$ for the first term and the inequality

$$(a+b)c \leq \frac{1}{2}(a+b+c)^2 \text{ for the second}$$

$$\begin{aligned}
\lambda \int_{\Omega'} |\eta D\Delta^h u|^2 &\leq \frac{1}{\epsilon} c^2 (\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)} + \|D\eta\Delta^h u\|_{L^2(\Omega')})^2 + \epsilon \|(\Delta^h u)\|_{L^2(\Omega')}^2 \\
&\quad + c(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)} + \|D\eta\Delta^h u\|_{L^2(\Omega')})^2.
\end{aligned}$$

Choose any $0 < \epsilon < \lambda/2$. Then subtract the second term on the first line of the RHS from the LHS to get

$$\begin{aligned}
\|\eta D\Delta^h u\|_{L^2(\Omega')}^2 &\leq c(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)} + \|D\eta\Delta^h u\|_{L^2(\Omega')})^2 \Rightarrow \\
\|\eta D\Delta^h u\|_{L^2(\Omega')} &\leq c(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)} + \|D\eta\Delta^h u\|_{L^2(\Omega')}) \\
&\leq c(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)} + \sup_{\Omega} |D\eta| \cdot \|\Delta^h u\|_{L^2(\Omega')}) \\
&\leq c(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)}) \cdot (1 + \sup_{\Omega} |D\eta|),
\end{aligned}$$

since $\|\Delta^h u\|_{L^2(\Omega)} \leq \|Du\|_{L^2(\Omega)} \leq \|u\|_{W^{1,2}(\Omega)} \leq \|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)}$ where we have applied the first Lemma to $u \in W^{1,2}(\Omega)$. Now we are done as we can choose η such that first $\eta|_{\Omega'} = 1$ (for the LHS !) and second $|D\eta| \leq \text{dist}(\Omega', \partial\Omega)$ (for the RHS) and so

$$\|1 \cdot D\Delta^h u\|_{L^2(\Omega')}^2 \leq c(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)}),$$

independently of h . So by our second Lemma the uniform boundedness of the difference quotients of Du in $L^2(\Omega')$ implies $Du \in W^{1,2}(\Omega') \Rightarrow u \in W^{2,2}(\Omega')$ and we have the desired estimate for its $W^{2,2}(\Omega')$ norm by the above inequality combined with the Lemma. \blacksquare

Now that $u \in W^{2,2}(\Omega')$ then the our original equation holds in the *usual* sense

$$Lu = a^{ij}D_{ij}u + D_i a^{ij}D_j u + b^i D_i u + c \cdot u = f,$$

a.e !